Almost Classical Skew Bracoids and the Yang-Baxter Equation

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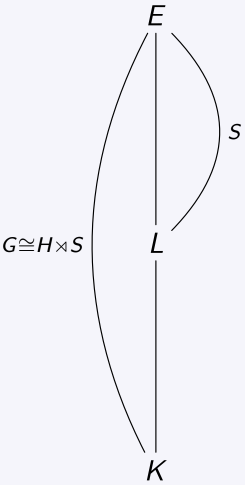
Aim

Skew bracoids that correspond to Hopf-Galois structures on almost classical extensions lead to solutions to the Yang-Baxter equation in the manner outlined earlier.



Question

What is special about these solutions?

Outline

1. Fundamental Definitions and Examples
2. Almost classical skew bracoids
3. The γ-function and solutions to the Yang-Baxter Equation

Outline

Skew braces and bracoids

Definition

A *skew (left) brace* is a triple $(G, \star, \cdot)$, where $(G, \star)$ and $(G, \cdot)$ are groups and for all $g, h, f \in G$

$$g \cdot (h \star f) = (g \cdot h) \star g^{-1} \star (g \cdot f).$$

Definition

A *skew (left) bracoid* is a 5-tuple $(G, \cdot, N, \star, \odot)$, where $(G, \cdot)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of G on N for which

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu),$$

for all $g \in G$ and $\eta, \mu \in N$.

Housekeeping

- We will frequently write $(G, N, \odot)$ or (G, N), for $(G, \cdot, N, \star, \odot)$.
- We will refer to $(N, \star)$ as the additive group and $(G, \cdot)$ as the multiplicative or acting group.
- Any identity will be denoted e, possibly with a subscript.

For example

Examples

- If $(G, \cdot)$ is a group then $(G, \cdot, \cdot)$ and $(G, \cdot^{op}, \cdot)$ are skew braces, the so-called *trivial* and *almost trivial* skew braces on G.
- Any skew brace $(G, \star, \cdot)$ can be thought of as a skew bracoid $(G, \cdot, G, \star, \odot)$, where $\odot$ is simply $\cdot$. If (G, N) is a skew bracoid with $\mathrm{Stab}_G(e_N)$ trivial we say that (G, N) is *essentially a skew brace*.
- For any group G we have the skew bracoid $(G, \{e\}, \odot)$ where of course the action $\odot$ is trivial.

Something more concrete

Examples

- Let $d, n \in \mathbb{N}$ such that $d|n$. Take $G = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_{2n}$ and $N = \langle \eta \rangle \cong C_d$. Then we get a skew bracoid $(G, N, \odot)$ for $\odot$ given by

$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}.$$

Outline

Containing a brace

Definition

We say that a skew bracoid (G, N) *contains a brace* if the subgroup $S = \text{Stab}_G(e_N)$ has a complement H in G, so that $G = HS$.

This is equivalent to saying that G contains a subgroup H for which (H, N) is essentially a skew brace.

Definition

A skew bracoid (G, N) is *almost a skew brace* if the subgroup $S = \text{Stab}_G(e_N)$ has a normal complement H in G, so that $G = HS \cong H \rtimes S$.

Almost classical skew bracoids

Definition

A skew bracoid (G, N) is *almost classical* if the subgroup $S = \mathrm{Stab}_G(e_N)$ has a normal complement H in G, and when thought of as a skew brace, (H, N) is trivial. Hereafter we will say that such a (H, N) is *essentially trivial*.

This is saying that when the operation in H is transferred to N, via $h \mapsto h \odot e_N$, it coincides with the original operation in N. Explicitly this means, for all $h_1, h_2 \in H$

$$(h_1 \odot e_N) \star (h_2 \odot e_N) = h_1 h_2 \odot e_N,$$

and consequently

$$(h_1 \odot e_N)^{-1} = h_1^{-1} \odot e_N.$$

Members of our Dihedral Cyclic Family

Example

Consider $(G, N) \cong (D_{2n}, C_d)$, using $r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}$. Then $S = \mathrm{Stab}_G(e_N) = \langle r^d, s \rangle$ since $r^i s^j \odot e_N = \eta^i$.

	n	d	S	Contains	Is Almost	Almost Classical
(1)	24	4	$\langle r^4, s \rangle$	✗	✗	✗
(2)	12	6	$\langle r^6, s \rangle$	$\langle r^4, rs \rangle$	✗	✗
(3)	12	4	$\langle r^4, s \rangle$	$\langle r^6, rs \rangle$		$\langle r^3 \rangle$
(4)	n	n	$\langle s \rangle$			$\langle r \rangle$
(5)	n even	n	$\langle s \rangle$		$\langle r^2, rs \rangle$	$\langle r \rangle$

Outline

The γ-function

Definition/Proposition

Given a skew bracoid $(G, \cdot, N, \star, \odot)$, we define the map $\gamma : (G, \cdot) \to \mathrm{Perm}(N, \star)$, sending g to γ_g, by

$$\gamma_g(\eta) = (g \odot e_N)^{-1} \star (g \odot \eta),$$

for $g \in G$ and $\eta \in N$.

Then γ is in fact a homomorphism, with image in $\mathrm{Aut}(N, \star)$. We call this map the γ-*function* of the skew bracoid.

Recall that this is the backbone of the route from a skew bracoid to a solution to the Yang-Baxter equation.

The γ-function of an almost classical skew bracoid

Let (G, N) be an almost classical skew bracoid with H such that $G \cong H \rtimes S$ and (H, N) is essentially a trivial skew brace.

Let $h, h_\eta \in H$ and $s \in S$ then,

$$\begin{aligned}\gamma_{hs}(h_\eta \odot e_N) &= (hs \odot e_N)^{-1} \star (hs \odot (h_\eta \odot e_N)) \\ &= (h \odot e_N)^{-1} \star (hsh_\eta \odot e_N) \\ &= (h^{-1} \odot e_N) \star (hsh_\eta s^{-1} \odot e_N) \\ &= sh_\eta s^{-1} \odot e_N.\end{aligned}$$

So we conjugate by the S part and the H part acts trivially.

Example

In the $G = \langle r, s\rangle \cong D_{2n}$ acting on $N = \langle \eta \rangle \cong C_n$ example, using $R = \langle r \rangle$, we have $\gamma_{r^i s^j}(\eta^k) = s^j r^k s^{-j} \odot e_N = \eta^{(-1)^j k}$.

The Yang-Baxter Equation

Definition

A *solution to the set-theoretic Yang-Baxter equation* (hereafter simply a *solution*) is a non-empty set G, together with a map $\boldsymbol{r} : G \times G \to G \times G$ satisfying

$$(\boldsymbol{r} \times 1)(1 \times \boldsymbol{r})(\boldsymbol{r} \times 1) = (1 \times \boldsymbol{r})(\boldsymbol{r} \times 1)(1 \times \boldsymbol{r})$$

as functions on $G \times G \times G$.

Given a solution $\boldsymbol{r}$ on G, for all $x, y \in G$ we write

$$\boldsymbol{r}(x, y) = (\lambda_x(y), \rho_y(x));$$

so that we have family of maps $\lambda_x : G \to G$ and a family of maps $\rho_y : G \to G$.

Properties of the Solution

Suppose G with $\boldsymbol{r}$ is a solution and write $\boldsymbol{r}(x, y) = (\lambda_x(y), \rho_y(x))$.
We say this solution is:

- *bijective* if $\boldsymbol{r}$ is bijective;
- *left non-degenerate* if λ_x is bijective for all $x \in G$;
- *right non-degenerate* if ρ_y is bijective for all $y \in G$;
- *non-degenerate* if $\boldsymbol{r}$ is both left and right non-degenerate.

Solutions from skew bracoids

Let (G, N) be a skew bracoid that contains a brace (H, N). We have that the map $a : h \mapsto h \odot e_N$ is a bijection between H and N, we write b for its inverse. Recall that with this we can define

$$\lambda_x(y) = b(\gamma_x(y \odot e_N))$$

and then

$$\rho_y(x) = \lambda_x(y)^{-1}xy,$$

for all $x, y \in G$.

This λ and ρ form a Lu-Yan-Zhu pair so that G with $\boldsymbol{r}(x, y) = (\lambda_x(y), \rho_y(x))$ forms a (right non-degenerate but possibly left degenerate) solution.

A matched product

Given this setup, we have that

- H with $\boldsymbol{r}$ is a bijective non-degenerate solution - the one coming from the (essentially a) skew brace (H, N);
- and restricting to S we have

$$\lambda_{s_1}(s_2) = b(\gamma_{s_1}(s_2 \odot e_N)) = e_G, \qquad \rho_{s_2}(s_1) = s_1 s_2.$$

  for $s_1, s_2 \in S$, so we get an entirely left degenerate sub-solution - if you like, the one coming from the skew bracoid $(S, \{e\})$.

In general, the solution on G is a matched product of these two sub-solutions. This via $\alpha : S \to \mathrm{Perm}(H)$ and $\beta : H \to \mathrm{Perm}(S)$ given by $\alpha_h(s) = (\rho_{h^{-1}}(s^{-1}))^{-1}$ and $\beta_s(h) = \lambda_s(h)$.

Solutions from almost a skew brace

If the skew bracoid (G, N) is almost a skew brace, so our complement H to S is normal in G, the actions $\alpha : S \to \mathrm{Perm}(H)$ and $\beta : H \to \mathrm{Perm}(S)$ are transparently the actions of S on H and H on S within G.

For $s \in S$ and $h \in H$ we have,

$$\beta_s(h) = \lambda_s(h) = b((s \odot e)^{-1}(s \odot (h \odot e))) = b(sh \odot e) = shs^{-1}.$$

and

$$\begin{aligned}
\alpha_h(s) &= (\rho_{h^{-1}}(s^{-1}))^{-1} \\
&= (\lambda_{s^{-1}}(h^{-1})^{-1}s^{-1}h^{-1})^{-1} \\
&= ((s^{-1}h^{-1}s)^{-1}s^{-1}h^{-1})^{-1} \\
&= (s^{-1}hss^{-1}h^{-1})^{-1} \\
&= s.
\end{aligned}$$

Almost classical solutions

Suppose (G, N) is almost classical due to a subgroup H of G, i.e. (H, N) is essentially trivial. Taking $h_1, h_2 \in H$ and $s_1, s_2 \in S$, the solution arising from (G, N) is the given by

$$\begin{aligned}\lambda_{h_1 s_1}(h_2 s_2) &= b(\gamma_{h_1 s_1}(h_2 \odot e_N)) \\ &= b(s_1 h_2 s_1^{-1} \odot e_N) \\ &= s_1 h_2 s_1^{-1}, \\ \rho_{h_2 s_2}(h_1 s_1) &= s_1 h_2^{-1} s_1^{-1} h_1 s_1 h_2 s_2.\end{aligned}$$

Note that restricting to H we recover the solution given by

$$\lambda_{h_1}(h_2) = h_2, \qquad \rho_{h_2}(h_1) = h_2^{-1} h_1 h_2,$$

which is a solution coming from the group H.

Almost Classical Running Example

Example

In our $(G, N) \cong (D_{2n}, C_n)$ example taking the complement $R = \langle r \rangle$ we have

$$\begin{aligned}\lambda_{r^i s^j}(r^k s^\ell) &= s^j r^k s^{-j} \\ &= r^{(-1)^j k},\end{aligned}$$

and

$$\begin{aligned}\rho_{r^k s^\ell}(r^i s^j) &= s^j r^{-k} s^{-j} r^i s^j r^k s^\ell \\ &= r^{-(-1)^j k + i + (-1)^j k} s^{j+\ell} \\ &= r^i s^{j+\ell}.\end{aligned}$$

Hence G with $\boldsymbol{r}(r^i s^j, r^k s^\ell) = (r^{(-1)^j k}, r^i s^{j+\ell})$ is a solution.

Almost a skew brace Running Example

Example

Suppose now that n is even and take the complement $H = \langle r^2, rs \rangle$ to S in $G \cong D_{2n}$.

It is convenient to transfer the operation from N onto H. To do this note that

$$b(\eta) = rs, \;\; b(\eta^2) = r^2, \;\; b(\eta^3) = r^3 s, \;\; b(\eta^4) = r^4, \; \dots$$

so we may think of H as a subgroup of $C_n \times C_2$ with rs as a generator.

Then the action of G on H can be thought of as $r^i s^j \odot (rs)^k = (rs)^{i+(-1)^j k}$, essentially as before.

Almost a skew brace Running Example

Example

Then with the skew bracoid written (G, H) we have,

$$\begin{aligned}
\lambda_{r^i s^j}(r^k s^\ell) &= b(\gamma_{r^i s^j}((rs)^k)) \\
&= b((rs)^{(-1)^j k}) \\
&= b(r^{(-1)^j k} s^{(-1)^j k}) \\
&= r^{(-1)^j k} s^k, \\
\rho_{r^k s^\ell}(r^i s^j) &= \lambda_{r^i s^j}(r^k s^\ell)^{-1} r^i s^j r^k s^\ell \\
&= s^k r^{-(-1)^j k} r^{i+(-1)^j k} s^{j+\ell} \\
&= r^{(-1)^k i} s^{j+k+\ell}.
\end{aligned}$$

Hence G with $\boldsymbol{r}(r^i s^j, r^k s^\ell) = (r^{(-1)^j k} s^k, r^{(-1)^k i} s^{j+k+\ell})$ is a solution.

Reduction in solutions

Consider the subgroup $K = \ker(\odot)$ of G, this is a subgroup of S. Simply restricting to K, we get the relations are the same as in S, meaning K with $\boldsymbol{r}$ is also a solution. But moreover, for $k \in K$ and all $hs \in G$, we have

$$\lambda_k(hs) = h, \qquad \rho_{hs}(k) = h^{-1}khs,$$
$$\lambda_{hs}(k) = e_G, \qquad \rho_k(hs) = hsk.$$

Proposition

If (G, N) contains a brace (resp. is almost a brace or almost classical) then the reduced form $(G/K, N)$ is contains a brace (resp. is almost a brace or almost classical).

Open Questions

- What does all this mean for the study of solutions using skew bracoids?
- I didn't give the solutions coming from the non-reduced skew bracoids, partially because they were a pain - especially in general - am I (still) missing something?
- How is the solution on a skew bracoid related to the solution on its reduced form?

Thank you for your attention!